Functional correctness of an optimized modular inversion algorithm

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2026-07-26 @ ITP'26 The talk will be uploaded soon @ tvallejos.cl

Context

Modular inversion

> For a prime m and $y \neq 0$, compute y^{-1}, such that, $yy^{-1} = 1 (\mathrm{mod}\, m)$.

Intensively used in cryptographic algorithm families, e.g. RSA and ECC

- **Efficiency** matters: large m (e.g. $2^{255} - 19$)
- **Security** constraints: constant time

In 2020 Pornin publishes an algorithm with ~25% perf gain w.r.t. best implem Bernstein and Yang (2019).
Used in:

1. **Falcon**: Post-quantum signature scheme (cf NIST PQ Cryptography Standardization)
2. **blst**: Library for performant signature schemes

Modular inversion via EGCD

```
mod_inv(y, m) :=
a <- y, b <- m, u <- 1, v <- 0
while a ≠ 0
    quotient <- b div a
    (b, a) <- (a, b - quotient * a)
    (v, u) <- (u, v - quotient * u)
return v
```

$$yy^{-1} = 1 (\mathrm{mod}\, m)$$

$$yy^{-1} + mu = 1 (\mathrm{mod}\, m)$$

$$yy^{-1} + mu = \gcd(y, m)$$

(Bezout's identity!)

A highly optimized modular inversion via EGCD

Algorithm 2 Extended Binary GCD (optimized algorithm)

Require: Odd modulus m ($m \geq 3$, $m \bmod 2 = 1$), value y ($0 \leq y < m$), and $k > 1$
Ensure: $1/y \bmod m$ (if $\mathrm{GCD}(y, m) = 1$)

$a \leftarrow y, u \leftarrow 1, b \leftarrow m, v \leftarrow 0$
for $1 \leq i \leq \lceil (2\mathrm{len}(m) - 1)/k \rceil$ **do**
  $n \leftarrow \max(\mathrm{len}(a), \mathrm{len}(b), 2k)$
  $\bar{a} \leftarrow (a \bmod 2^{k-1}) + 2^{k-1}\lfloor a/2^{n-k-1} \rfloor$ ▷ $\bar{a} < 2^{2k}$
  $\bar{b} \leftarrow (b \bmod 2^{k-1}) + 2^{k-1}\lfloor b/2^{n-k-1} \rfloor$ ▷ $\bar{b} < 2^{2k}$
  $f_0 \leftarrow 1, g_0 \leftarrow 0, f_1 \leftarrow 0, g_1 \leftarrow 1$
  for $1 \leq j \leq k - 1$ **do**
    if $\bar{a} = 0 \bmod 2$ **then**
      $(f_0, g_0) \leftarrow (f_0 - f_1, g_0 - g_1)$
    $(f_1, g_1) \leftarrow (2f_1, 2g_1)$
  $(a, b) \leftarrow ((af_0 + bg_0)/2^{k-1}, (af_1 + bg_1)/2^{k-1})$
  if $a < 0$ **then**
    $(a, f_0, g_0) \leftarrow (-a, -f_0, -g_0)$
  if $b < 0$ **then**
    $(b, f_1, g_1) \leftarrow (-b, -f_1, -g_1)$
  $(u, v) \leftarrow (uf_0 + vg_0 \bmod m, uf_1 + vg_1 \bmod m)$
$v \leftarrow v/2^{(k-1)\lceil (2\mathrm{len}(m)-1)/(k-1) \rceil} \bmod m$
if $b \neq 1$ **then**
  return 0 (value y is not invertible)

Note. An initial version of this note used a slightly more aggressive algorithm ( ) which was wrong.

Binary EGCD

```
bin_mod_inv(y, m   ) :=
a <- y, b <- m, u <- 1, v <- 0,
while a ≠ 0
    if a = 0 mod 2
        a <- a / 2
        u <- u / 2 mod m
    else
        if a < b
            swap(a,b)
            swap(u,v)
        a <- (a - b) / 2
        u <- (u - v) / 2 mod m
return v
```

Observe

- No decision is made on values `u` and `v`
- Parity needs only lowest bit
- `a < b` needs a sensible amount of higher bits

Possible optimizations?

- Do batch update of `u` and `v`
- Compute on approximations of `a` and `b`

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- Do batch update of `u` and `v`
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Batch update of u and v

```
opt_mod_inv (y, m, k) :=
a <- y, b <- m, u <- 1, v <- 0          // init variables and coeff for a and b
while a ≠ 0
```

Batch update of u and v

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opt_mod_inv (y, m, k) :=
a ← y, b ← m, u ← 1, v ← 0          // init variables and coeff for a and b
while a ≠ 0
    R                <- init_record()// records updates during execution
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Approximation-based optimization

Approximation based modular inversion

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    R              <- init_record()// records updates during execution
    (α,β)          <- approx(a,b,k)
    do k iters

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Desynchronization

(1) have a, α odd, and $\alpha \geqslant \beta$, but $a < b$.
(2) have `α' := (α - β)/2` , and `a' := (a-b)/2` .
(3) have `α'` , and `a'` odd, and `α' < β`

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$a \leftarrow a'$ ($\mathbf{a} < \mathbf{0}$!!)

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$\alpha < \beta$ (swap)

`α ← (α - β)/2`

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$a < b$

a ← a' (**a < 0 !!**)

`a'` is odd

`a' < b` (**sync**)

`a'' ← (b - a')/2`

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`α'` is odd

$\alpha < \beta$ (swap)

$\alpha \leftarrow$ `(α - β)/2`

a is odd

$a < b$

$a \leftarrow a'$ ($a < 0$!!)

`a'` is odd

`a'` < b (**sync**)

`a'' ← (b - a')/2`

but `a' < 0`

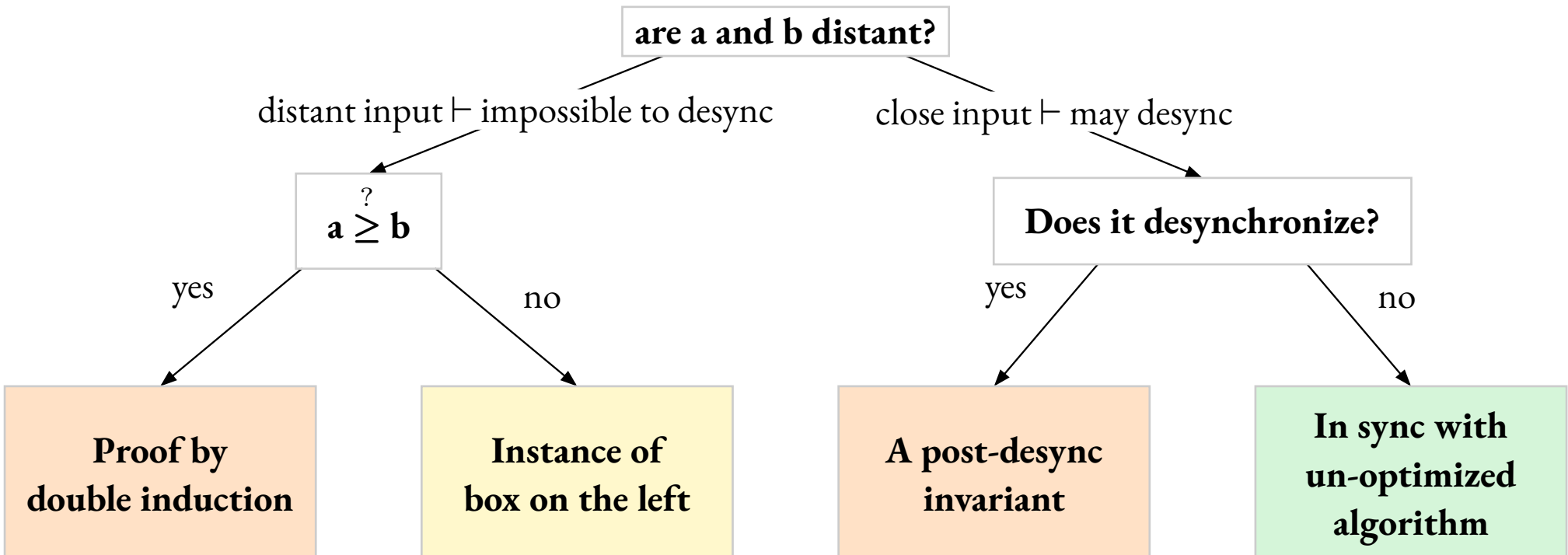
`a'' = (b + |a'|)/2`

`a' < b + a'`

Functional correctness

Why is using approximations correct?

Goal: show `a` converges to `0`, but after desynchronization `a - b` may increase.
Strategy: establish a contract for the approximation loop. After k iterations at least k bits have been lost.



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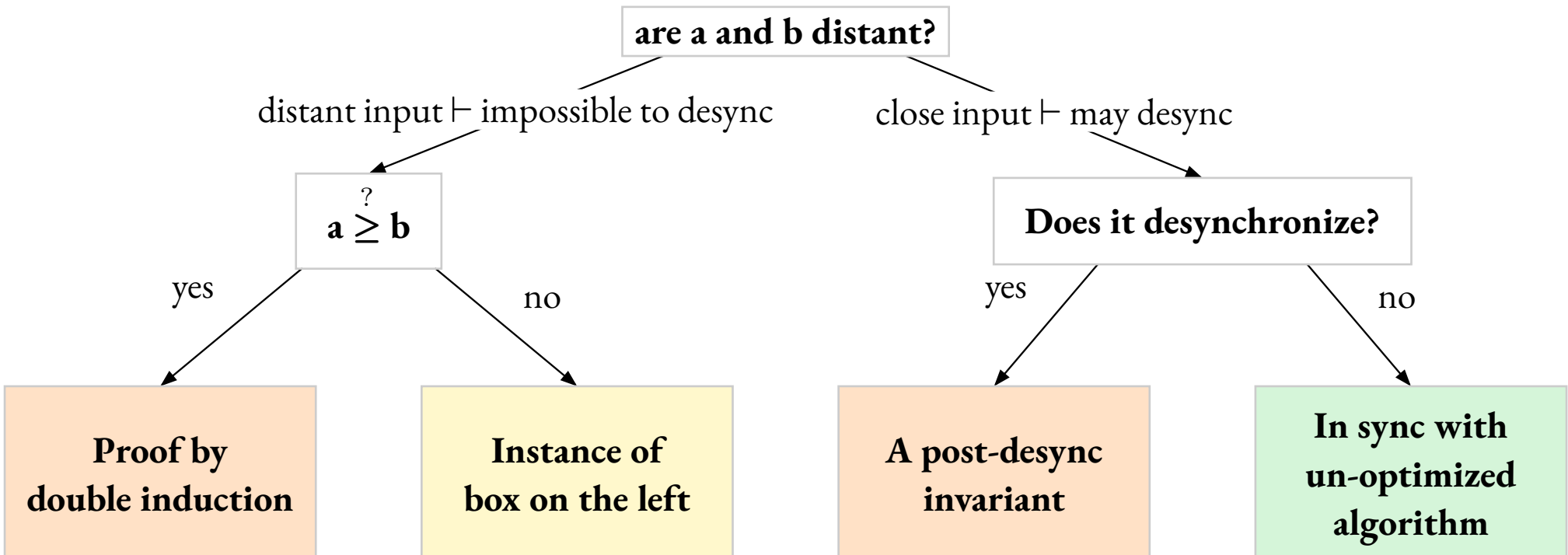

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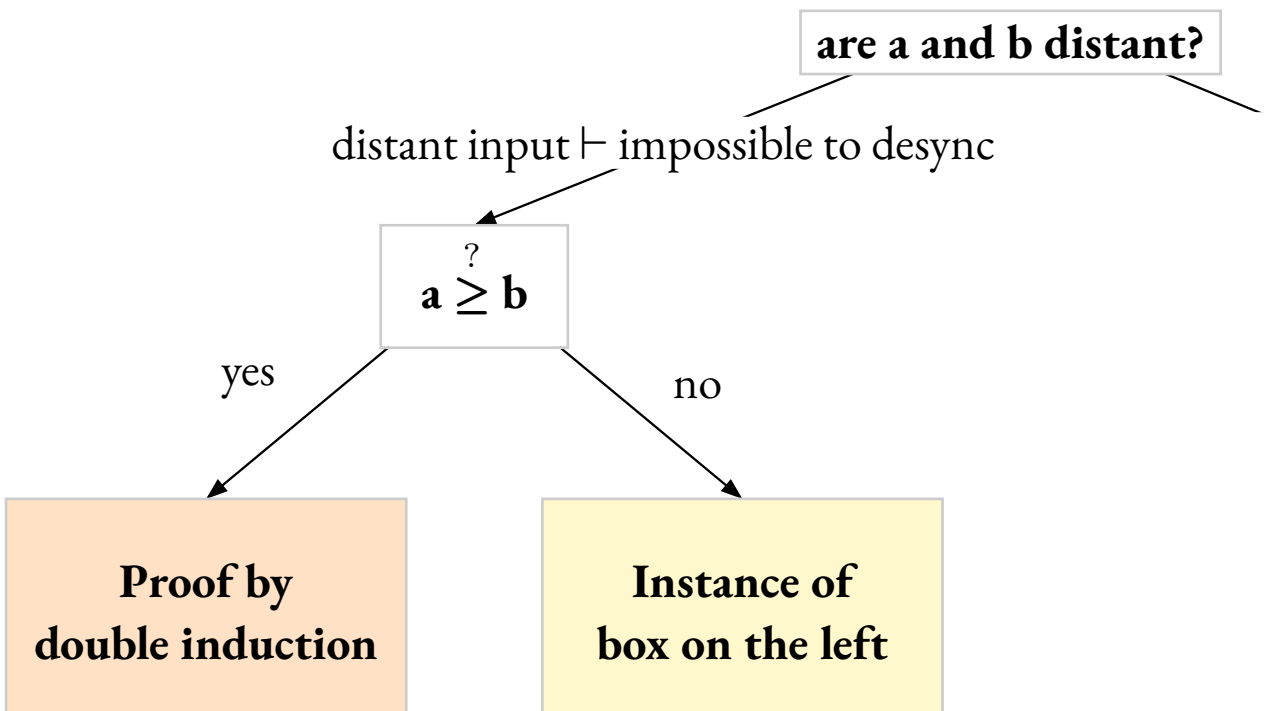

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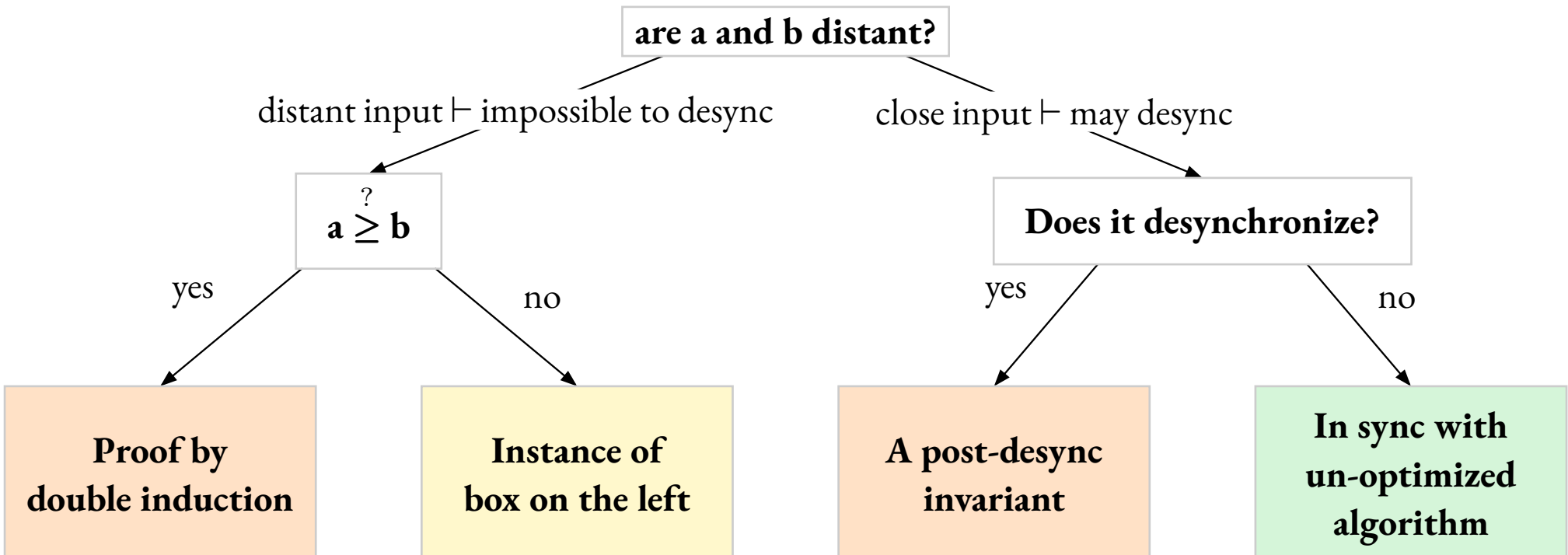


Desynchronization proof sketch

Let $\ell(x)$ be the length in bits of x.
Under hypotheses:

- $\ell(a)$ and $\ell(b)$ are close
- Desynchronization at iteration d

Then, after k iterations $\ell(a) + \ell(b)$ decreases at least k bits.

Proof sketch.

1. At iteration d, at least d bits were lost.
2. At iteration $d + 1$, k bits were lost.
3. $H(t) \triangleq P(t) \vee Q(t)$ is an invariant post-desync, and preserves k bits were lost.

With

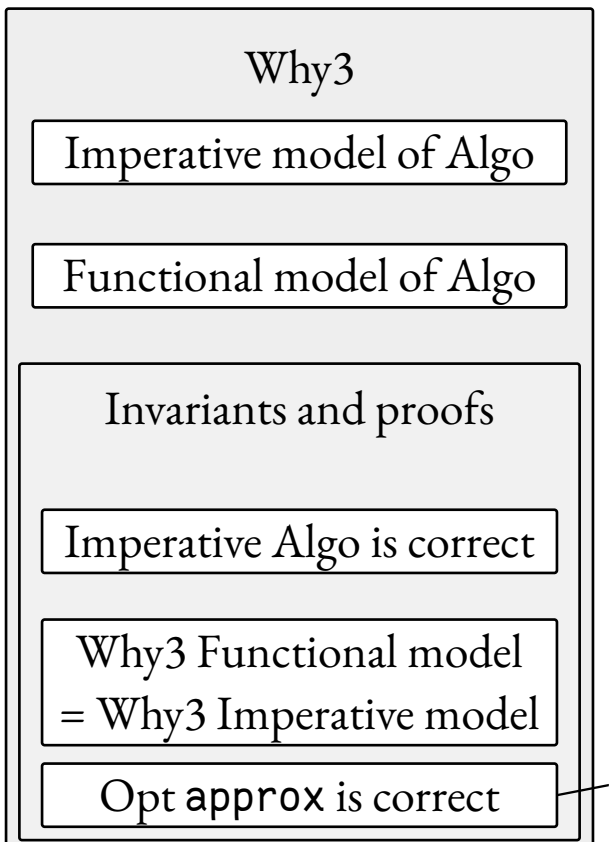
- $P(t) \triangleq M_t \geq 2^{n-k+1} \wedge -2^{n-k} < m_t \leq 0$
- $Q(t) \triangleq \max(|a_t|, |b_t|) < 2^{n-k+1}$
- $M_t \triangleq \max(a_t, b_t) \quad \wedge \quad m_d \triangleq \min(a_t, b_t) \quad a_t \triangleq$ `batch_update(a,b,R,t)` $\quad b_t \triangleq \ldots$

Mechanization

Collaboration between Why3 and Rocq

- Initial plan: a complete formalization in Why3
- Implement the binary EGCD algorithm in Why3
- Iterative refinement of optimizations/proofs
- At approximation optimization:
  - Context too big for Why3+SMT to handle
  - Lack of expressivity to state and prove theorems
- Rocq model and proof that approximation optimization is correct
- Connect the proof via a bridge between Why3 and Rocq

Overview of the mechanization



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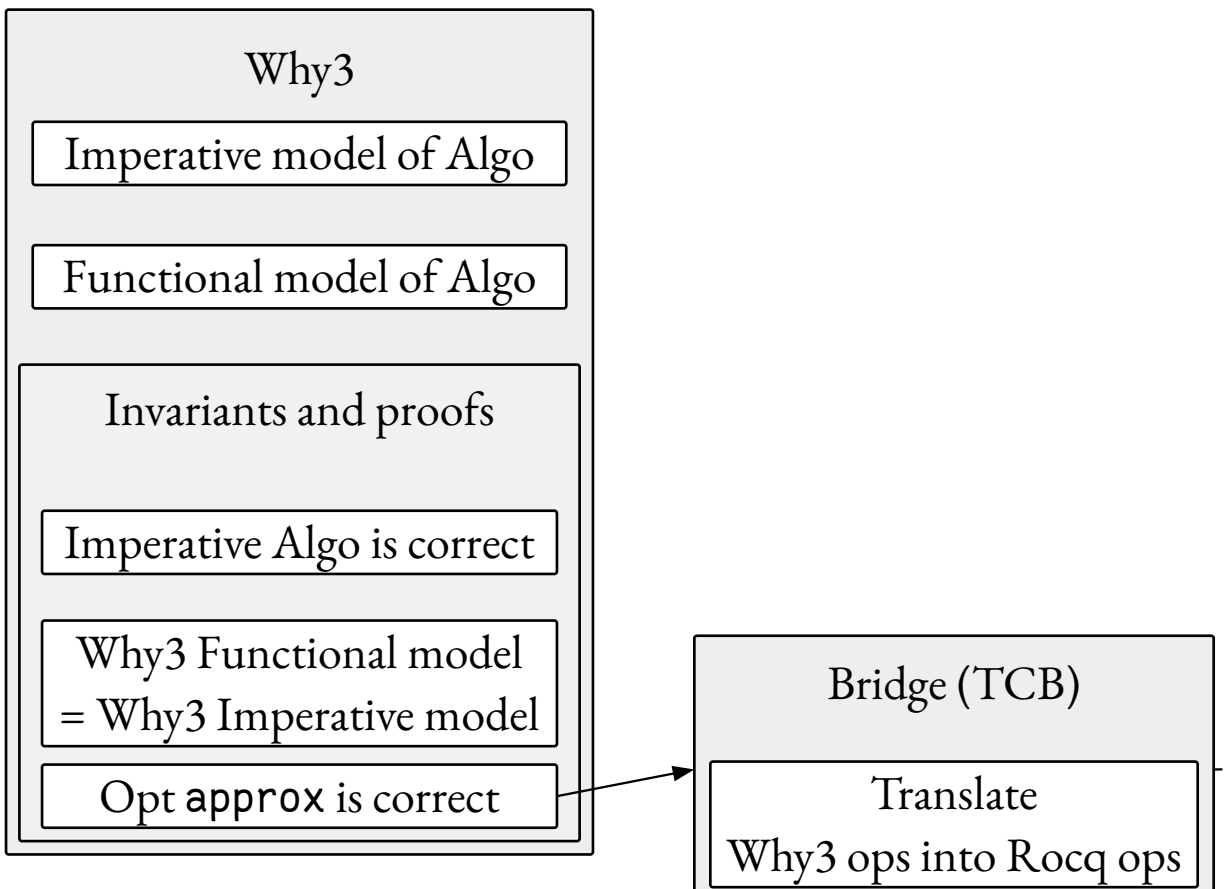

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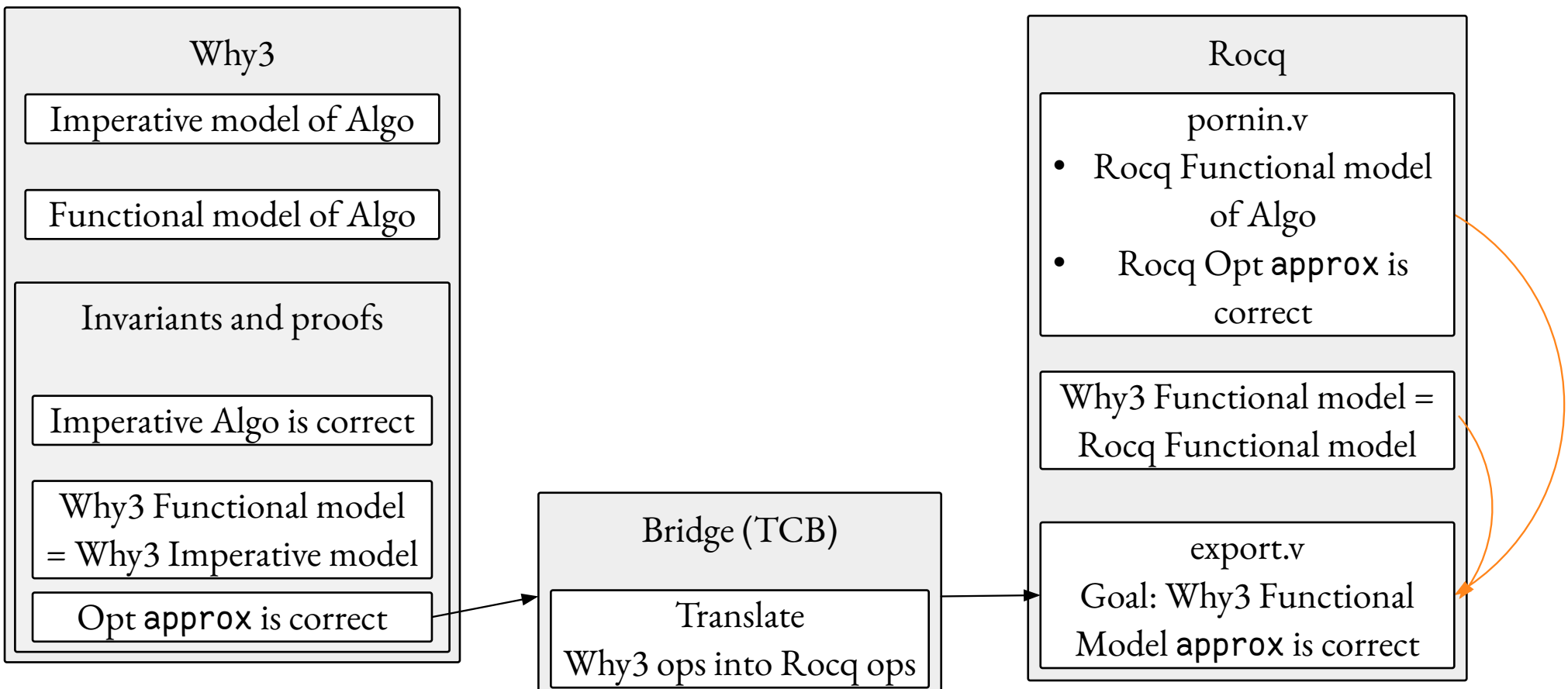

The bright side

- Near zero-effort spent glueing Rocq and Why3 models
- Some automation in Rocq for linear integer arithmetic
- No issue in mechanizing double induction arguments in Rocq

The not so bright side

- SMT solvers were not helpful with non-linear arithmetic
- Floyd-Hoare logic was not suitable to deal with desynchronization proofs
- Tons of proof transferring from $\mathbb{N}$ to $\mathbb{Z}$ (e.g. lift `e1 < e2` from `nat` to `int`)

Summary/Context

- Highly optimized algorithm used in critical security context
- Possibly incorrect functional correctness proof

Contributions

- Complete functional correctness proof
- Mechanization of the functional correctness proof
- Verification of an imperative implementation of the algorithm

The current version of the algorithm, and the revised proof, are believed correct.

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